Series 1

Exercise 1

The goal of this exercise is to derive the expression of the Green's function $g(\boldsymbol{r},\boldsymbol{r}')$ of the Helmholtz equation. As shown in class, this function satisfies the differential equation:

$$\nabla^2 g(\boldsymbol{r},\boldsymbol{r}') + k^2 g(\boldsymbol{r},\boldsymbol{r}') = -\delta(\boldsymbol{r}-\boldsymbol{r}')$$

1) In one sentence, give a good physical reason to justify that $g(\boldsymbol{r},\boldsymbol{r}')$ is actually a function of $\boldsymbol{r}-\boldsymbol{r}'$. Namely, $g(\boldsymbol{r},\boldsymbol{r}') = g(\boldsymbol{r}-\boldsymbol{r}')$.
2) In one sentence, give a good physical reason to justify that $g(\boldsymbol{r},\boldsymbol{r}')$ is actually a function of $|\boldsymbol{r}-\boldsymbol{r}'|$. Namely, we can go even further and write $g(\boldsymbol{r},\boldsymbol{r}') = g(|\boldsymbol{r}-\boldsymbol{r}'|)$.

We will now note $r_1 = |\boldsymbol{r}-\boldsymbol{r}'|$

3) By working in a spherical coordinate system, show that g satisfies

$$\frac{1}{r_1^2}\frac{d}{dr_1}\left[r_1^2\frac{dg}{dr_1}\right] + k^2 g = -\delta(r_1)$$

You may directly use the expression of ∇^2 found online at your favorite website. The sigularity of this equation at $r_1 = 0$ will be handled in a special way.

4) When $r_1 \neq 0$, show that the equation in step 3) reduce to:

$$\frac{d^2}{dr_1^2}[r_1 g] + k^2[r_1 g] = 0$$

5) Show that the solution at $r_1 \neq 0$ is therefore:

$$g(r_1) = C\ \frac{e^{-jkr_1}}{r_1}$$

where C is a constant to be determined.

6) The constant can be calculated by integrating the equation of step 3 over a spherical volume of radius ε, and taking the limit $\varepsilon \to 0$.
    a. Start with the right hand side. This is easy is you remember the definition of the Dirac distribution.
    b. Work on the left hand side and integrate over spherical shells of volume $4\pi r_1^2\, dr_1$, between 0 and ε. The singularity at $r_1 = 0$ disappears and the result after taking $\varepsilon \to 0$ should give:

$$C = 1/4\pi$$

For those interested, try to find the Green's function of the 2D Helmholtz equation. This is another story !

Exercise 2

We would like to calculate the exact electromagnetic field radiated by an infinitesimal dipole described by the current distribution:

$$\boldsymbol{J}(\boldsymbol{r}') = \boldsymbol{z}\, I\, L_0 \delta(\boldsymbol{r}')$$

where I is a current in Ampères and L_0 an effective length parameter.

1) Check that the unit of this current distribution is indeed A/m^2.
2) Using the formulas on the last slide of the first course, compute the vector potential.
3) Calculate the exact magnetic field $\boldsymbol{H}(\boldsymbol{r})$.
   Hint: Formulas like $\nabla \times (f\boldsymbol{a}) = \nabla f \times \boldsymbol{a} + f\nabla \times \boldsymbol{a}$ can allow you to do the calculation fast.
4) Calculate the exact electric field $\boldsymbol{E}(\boldsymbol{r})$.
   Hint: Two ways to go. Maxwell's equations or via ϕ. In case of emergency, stay seated, remain calm, and follow the flight attendants' instructions.
5) Can you identify in these fields which part is called the near field, which part is called the far field, and why ?